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APPLICATION OF BOUNDARY-BORDER PROBLEMS FOR THE ANALYSIS OF THE STATE OF COMPLEX SYSTEMS

When calculating the values of the objective function and its parameters for complex systems, it remains the most time-consuming to obtain an analytical form of the objective function. This is explained by the fact that the analytical solution exists only for single-layer materials, homogeneous in terms of their geometric parameters and physical characteristics, on which load sources act. The correctness of the boundary-value problems describing the state of the above-mentioned systems is substantiated by well-known methods from the traditional theory of the existence and unity of the solution of boundary-value problems. In practice, however, it is quite often necessary to calculate the value of the objective function for a heterogeneous material under the influence of load sources of physical fields with different modes of action. It is possible to obtain the solution of boundary value problems either by determining additional conditions of existence and unity of the solution of boundary value problems, or by averaging the values of the parameters of the objective function and the corresponding system of constraints on its parameters in order to reduce the constructed boundary value problem to a more classical form. This means that in the second case, the boundary value problem for the simulated process is replaced by such a boundary value problem for which there will be no doubts about its correctness. At the same time, the accuracy of the calculated values of the objective function and its parameters will decrease.

The article proposes a method for parameterizing the temperature field of a multilayer material under the influence of sources of thermal load. The correctness of the calculated mathematical model built in the article is substantiated by methods from the theory of determining correctness conditions and their application for a multipoint boundary value problem over the space of generalized functions in a multilayer environment. This will guarantee the correctness of not only the boundary value problem given in the article for the system of heterogeneous differential equations of heat conduction, but also applied optimization mathematical models.

Key words: *correctness, boundary value problems, complex systems, parameterization, multilayer environment.*

Formulation of the problem. To solve applied problems of optimization of technical systems, it is necessary to correctly formulate the initial boundary value problems that describe the state of the object under study under the influence of sources of physical fields. The features of the modeled system influence the type of boundary value problems and, as a result, the choice of methods for solving boundary value problems, as well as methods for optimizing the values of the goal function and its parameters. This task is most relevant in the case of a non-standard shape of the object under study and specific features of the technical means of influence.

The article proposes a method for constructing boundary value problems for thermal effects on multilayer materials and parameterizing the temperature field. Taking into account the nonlinearity of differential equations, restrictions on the values of the objective function and its parameters, the multiplicity of the solution function and, as a result, the nonlinearity of the objective function and the multiextremal nature of applied problems of optimizing the parameters of thermal effects, have a decisive influence on the choice of methods for solving the correctness of boundary value problems. The authors constructed a resolving function of solutions for a system of pseudodifferential equations in a multilayer medium and showed that on a segment of generalized functions this function will satisfy the parabolicity condition. Fulfillment of this condition will ensure the correctness of the boundary value problem presented in this article in the spaces of infinitely differentiable generalized functions bounded on a segment, and also guarantee the correctness of applied optimization mathematical models for searching and enumerating local extrema of the temperature field and its parameters.

Analysis of recent research and publications. Mathematical models and methods of their implementation are proposed in the article [1] for analyzing the state of information and communication networks and increasing the efficiency of their functioning during threats to information security at enterprises in Ukraine. The authors of the publication [2] applied simulation modeling methods to increase the level of information security in the computer environment. The correct solvability of a nonlocal multipoint boundary value problem for a separate evolutionary pseudo-differential equation is proved [3]. In the article [4], a controllability function for two-dimensional, three-dimensional, and four-dimensional systems is proposed, which takes into account their specific features. The authors of the article [5] proposed mathematical models and computational methods for forecasting the appearance of threats in the supply of electric energy, solving problems with the supply of energy to critical infrastructure objects. A deterministic model was developed to optimize the energy supply of industrial enterprises in urban conditions and the optimal operation of the local energy system [6]. In publications [7, 8] applied problems of optimization of ecological systems are solved in order to increase the level of quality of the ecological situation in a separate region of the country, fight against threats to the ecological situation, overcome possible consequences of threats to the ecological system in a separate region of Ukraine. The problem of geometric design has been solved, which consists in eliminating empty space when packing objects in order to reduce the cost of packaging material [9, 10]. In articles [11, 12], methods of economic and mathematical modeling are applied to stabilize the work of the banking system of Ukraine in the current conditions.

Task statement. To propose a method for constructing boundary value problems for multilayer objects under the influence of thermal loading sources depending on the characteristics of the simulated systems.

Outline of the main material of the study. Let us construct a system of differential equations from the boundary value problem of the process of thermal impact on a multilayer material:

$$\begin{cases} \rho_1 c_1 \dfrac{\partial T_1(z,t)}{\partial t} = \lambda_1 \Delta T_1 + q_1; \\ \rho_2 c_2 \dfrac{\partial T_2(z,t)}{\partial t} = \lambda_2 \Delta T_2 + q_2; \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \rho_N c_N \dfrac{\partial T_N(z,t)}{\partial t} = \lambda_N \Delta T_N + q_N, \end{cases} \qquad (1)$$

where $T_e(z,t)$ – temperature field in e-th layer of the multilayer material;

z – spatial coordinate;

z_e – distance from the center of the impact source to a point in e-th layer of multilayer material, in which the value of the temperature field is calculated;

t – exposure time;

t_e – time parameter.

ρ_e – density coefficient of distribution points of temperature fields in the e-th layer of material;

c_e – heat capacity coefficient of temperature field distribution points;

λ_e – thermal conductivity coefficient in e-th layer of the multilayer material.

When mathematical modeling, it is worth taking into account the boundary conditions of the beginning and end of the thermal effect on the multilayer (N – layer) material:

$$\begin{cases} T(z,t)\big|_{t=t_0}^{z=z_0} = T_0; \\ T(z,t)\big|_{t=t_N}^{z=z_N} = T_N, \end{cases} \qquad (2)$$

where T_0 – temperature of the material at the beginning of thermal exposure;

T_N – material temperature at the end of thermal exposure.

To take into account the multilayer structure of the material, we introduce the equalities between the media:

$$\begin{cases} T_1(z_1,t_1) = T_2(z_2,t_2), \quad -\lambda_1 \dfrac{\partial T_1}{\partial z} = -\lambda_2 \dfrac{\partial T_2}{\partial z}; \\ T_2(z_2,t_2) = T_3(z_3,t_3), \quad -\lambda_2 \dfrac{\partial T_2}{\partial z} = -\lambda_3 \dfrac{\partial T_3}{\partial z}; \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ T_{N-1}(z_{N-1},t_{N-1}) = T_N(z_N,t_N), \quad -\lambda_{N-1} \dfrac{\partial T_{N-1}}{\partial z} = -\lambda_N \dfrac{\partial T_N}{\partial z}, \end{cases} \qquad (3)$$

where T_e – temperature of points in the e-th layer of a multilayer material;

z_e – spatial parameter;

t_e – duration of heat exposure.

Boundary conditions of the 3rd kind:

$$\left(\lambda_1 \frac{\partial T_1}{\partial z} - A\left(T_1 - T_{ext}\right) \right)\bigg|_{z=0} = 0\,, \qquad (4)$$

where λ_1 – thermal conductivity coefficient of the outer layer of the material;
A – heat transfer parameter of the outer layer of material;
T_1 – outer layer temperature;
T_{ext} – ambient temperature where the material is placed.

Let us prove the correctness of the boundary value problem (1)–(4). To do this, consider a homogeneous:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = A_1\left(\frac{\partial}{i\partial x}\right)u(x,t), & t \in [0;t_1]; \\ \frac{\partial u(x,t)}{\partial t} = A_2\left(\frac{\partial}{i\partial x}\right)u(x,t), & t \in [t_1;t_2]; \\ \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \\ \frac{\partial u(x,t)}{\partial t} = A_n\left(\frac{\partial}{i\partial x}\right)u(x,t), & t \in [t_{N-1};T] \end{cases}, \quad (5)$$

with the condition

$$B_0\left(\frac{\partial}{i\partial x}\right)u(x,0) + B_1\left(\frac{\partial}{i\partial x}\right)u(x,t_1) + \dots + B_n\left(\frac{\partial}{i\partial x}\right)u(x,T) = \varphi(x) \quad (6)$$

and inhomogeneous boundary value problems:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = A_1\left(\frac{\partial}{i\partial x}\right)u(x,t) + f(x,t); \\ \frac{\partial u(x,t)}{\partial t} = A_2\left(\frac{\partial}{i\partial x}\right)u(x,t) + f(x,t); \\ \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \\ \frac{\partial u(x,t)}{\partial t} = A_n\left(\frac{\partial}{i\partial x}\right)u(x,t) + f(x,t) \end{cases} \quad (7)$$

with the condition

$$B_0\left(\frac{\partial}{i\partial x}\right)u(x,0) + B_1\left(\frac{\partial}{i\partial x}\right)u(x,t_1) + \dots + B_n\left(\frac{\partial}{i\partial x}\right)u(x,T) = 0, \quad (8)$$

where $A_k\left(\frac{\partial}{i\partial x}\right)$, $B_k\left(\frac{\partial}{i\partial x}\right)$ – pseudodifferential operators with symbols from the space of infinitely differentiable functions of power growth.

We will consider in Sobolev-Slobodetsky spaces H_l^s, as well as in their projective limit $S = \bigcap_{s,l} H_l^s$ – L. Schwartz space [13, 14].

Let us apply the Fourier transform (in spatial variables) to the equations from the homogeneous boundary value problem (5)–(6). We obtain a dual boundary value problem:

$$\begin{cases} \frac{\partial \tilde{u}(s,t)}{\partial t} = A_1(s)\tilde{u}(s,t); \\ \frac{\partial \tilde{u}(s,t)}{\partial t} = A_2(s)\tilde{u}(s,t); \\ \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \\ \frac{\partial \tilde{u}(s,t)}{\partial t} = A_n(s)\tilde{u}(s,t). \end{cases}, \quad (9)$$

with the condition:

$$B_0(s)\tilde{u}(s,0) + B_1(s)\tilde{u}(s,t_1) + \dots + B_n(s)\tilde{u}(s,T) = \varphi(s). \quad (10)$$

Similarly, applying the Fourier transform to equations (7)–(8), we obtain the following dual boundary value problem:

$$\begin{cases} \frac{\partial \tilde{u}(s,t)}{\partial t} = A_1(s)\tilde{u}(s,t) + f(s,t); \\ \frac{\partial \tilde{u}(s,t)}{\partial t} = A_2(s)\tilde{u}(s,t) + f(s,t); \\ \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \\ \frac{\partial \tilde{u}(s,t)}{\partial t} = A_n(s)\tilde{u}(s,t) + f(s,t). \end{cases}, \quad (11)$$

with the condition:

$$B_0(s)\tilde{u}(s,0) + B_1(s)\tilde{u}(s,t_1) + \dots + B_n(s)\tilde{u}(s,T) = 0. \quad (12)$$

Resolving function of the boundary value problem (9)–(10):

$$Q(s,t) = \begin{cases} \exp tA_1(s) / \Delta(s); \\ \exp tA_2(s) / \Delta(s); \\ \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \; \dots \\ \exp\left((t - t_{n-1})A_n(s) + \dots + t_1A_1(s)\right) / \Delta(s), \end{cases} \quad (13)$$

where

$$\Delta(s) = B_0(s) + \dots + B_n(s)\exp\begin{pmatrix} t_1A_1(s) + +(t_2 - t_1)A_2(s) + \\ + \dots + (T - t_{n-1})A_n(s) \end{pmatrix} \neq 0. \quad (14)$$

We found that the homogeneous boundary value problem (5)–(6) is correctly solvable in the space of generalized functions bounded on a segment when the resolving function is bounded by a power polynomial together with its derivatives to a fixed order, i.e. $\left|D^kQ(s,t)\right| \le C_k\left(1+|s|\right)^{p_k}$.

Note that if instead of the multipoint condition (6) we consider the two-point condition, then this boundary value problem may be incorrect.

Let us proceed to the consideration of the boundary value problem for the inhomogeneous equation (7)–(8). As shown in article [14], if for the dual boundary value problem (9)–(10) there is a resolving function $Q(s;t)$, then in problem (11)–(12) there is a Green's function. This means that a parabolic boundary value problem can be perturbed by a subordinate pseudodifferential operator and the following statement holds.

The parabolicity condition for the homogeneous boundary value problem (5)–(6) is necessary for correct solvability in space of infinitely differentiable generalized functions of the perturbed equation with homogeneous boundary conditions bounded on the segment under sufficiently small perturbations.

The results obtained by the authors make it possible to guarantee the correctness of not only the boundary value problem (1)–(4) constructed in this article, but also the correctness of boundary value problems for other thermophysical systems. In addition, due to the specifics of optimizing the parameters of thermal effects on multilayer materials [15, 16], the correctness of boundary value problems implies the correctness of applied optimization problems of searching for local extrema of the goal function.

Note that by applying the results obtained, we can guarantee the correctness for the applied optimization problem of improving the quality of preparation of biomaterial for laser segmentation. The presented mathematical model is an auxiliary model and is not associated with optimization of the vector of parameters of the thermal effect on the material. However, it is important from the point of view of high-quality implementation of the entire biotechnological process of laser division of microbiological material.

The essence of the problem under consideration is as follows. Microbiological material is usually stored frozen in liquid nitrogen. Before dividing it, it is necessary to defrost the material without tearing the tissue. This means that during the defrosting process it is necessary to control the appropriate parameters to guarantee acceptable thermal stress values. The uniformity of the final distribution of the temperature field in the volume of microbiomaterial Ω can be characterized by the following mathematical model:

$$\left(\max_{\substack{(x,y,z)\in\Omega^* \\ t\in[t_0;t^*]}} T(x,y,z,t) - \min_{\substack{(x,y,z)\in\Omega^* \\ t\in[t_0;t^*]}} T(x,y,z,t) \right) \to \min_{z^*\in Z}, \quad (15)$$

where z^* – vector of parameters of thermal impact on the material.

In this case, it is necessary to fulfill the restriction on the minimum and maximum values of the temperature field in the microbiomaterial Ω:

$$\begin{cases} T_1^* \le \max T \le T_2^*; \\ T_3^* \le \min T \le T_4^*, \end{cases} . \quad (16)$$

where T_1^* – specified minimum permissible value of the maximum temperature field;
T_2^* – specified maximum permissible value of the maximum temperature field;
T_3^* – specified minimum permissible value of the minimum temperature field;
T_4^* – specified maximum permissible value of the minimum temperature field.

Let us note that fulfilling the requirements of the mathematical model (15)–(16) will allow, on the one hand, to have high-quality source material, and on the other hand, it will make it possible to correctly formulate the corresponding boundary value problem underlying the construction of mathematical models of thermal effects on biomaterials.

Conclusions. The article proposes a computational mathematical model that describes the state of a multilayer material under the influence of thermal loading sources. In connection with the specific features of the system under study (multilayer material under thermal influence) and the peculiarities of thermal treatment modes to improve the accuracy of calculating the temperature of exposure and technical parameters of hardware, the authors paid special attention to determining and studying their applicability for the correctness conditions of this boundary value problem. To prove the correctness of the boundary value problem, specialized approaches are used, which consist of methods over the functional space of generalized distributions smoothly depending on time. The condition for the parabolicity of the resolving function of solutions in the space of generalized functions bounded on a segment, together with its derivatives up to a fixed order, is proved. This will allow us to transfer the obtained result to determine the conditions for the correct solvability of a number of boundary value problems that describe the state of thermophysical systems, whose correctness cannot be guaranteed by the traditional theory of the existence and uniqueness of boundary value problems due to the specific features of the differential symbol of the operator. In addition, the results obtained can be used to prove the correctness of applied optimization mathematical models for the process of thermal attraction on a heterogeneous material.

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Левкін Д.А., Завгородній О.І., Гулієва Д.О., Левкін А.В. ЗАСТОСУВАННЯ КРАЙОВИХ ЗАДАЧ ДЛЯ АНАЛІЗУ СТАНУ СКЛАДНИХ СИСТЕМ

Під час розрахунку значень функції мети і її параметрів для складних систем найбільш трудомістким залишається отримати аналітичний вид функції мети. Це пояснюється тим, що аналітичний розв'язок існує лише для одношарових, однорідних за своїми геометричними параметрами та фізичними характеристиками матеріалів, на які діють джерела навантаження. Коректність крайових задач, які описують стан вищезгаданих систем обґрунтована відомими методами з традиційної теорії існування та єдиності розв'язку крайових задач. На практиці ж, доволі часто потрібно розрахувати значення функції мети для неоднорідного матеріалу під дією джерел навантаження фізичних полів з різними режимами дії. Отримати розв'язок крайових задач можливо або за рахунок визначення додаткових умов існування та єдиності розв'язку крайових задач, або через усереднення значень параметрів функції мети і відповідної системи обмежень на її параметри з метою зведення побудованої крайової задачі до більш класичного виду. Це означає, що в другому випадку крайову задачу для модельованого процесу замінюють на таку крайову задачу, для якої не буде виникати сумнівів її коректність. При цьому зменшиться точність розрахованих значень функції мети і її параметрів.

В статті запропонований метод для параметризації температурного поля багатошарового матеріалу під дією джерел термічного навантаження. Коректність побудованої в статті розрахункової математичної моделі обґрунтована методами з теорії визначення умов коректності та їх застосування для багатоточкової крайової задачі над простором узагальнених функції в багатошаровому середовищі. Це дозволить гарантувати коректність не лише наведеної в статті крайової задачі для системи неоднорідних диференціальних рівнянь теплопровідності, а також і прикладних оптимізаційних математичних моделей.

Ключові слова: *коректність, крайові задачі, складні системи, параметризація, багатошарове середовище.*